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Solutions of Generalized Euler-Poisson-Darboux Equation and Their Properties

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ABSTRACT

The subject of this paper is the study of classical EPD equation in $m+1$ variables, which in its full generality has been investigated extensively. We consider the equation

$$\frac{\partial^2 u}{\partial t^2}+\frac{k}{t}\frac{\partial u}{\partial t}=\sum_{i=1}^{m}\frac{\partial^2 u}{\partial {x_i}^2}+\frac{\lambda_i}{{x_i}^2}u$$

Where k and λ_i are real or complex parameters and that $k\neq -1,-3.-5,\cdots\cdots$. We construct general solutions in an explicit form expressed by the Appell and Lauricella hypergeometric functions of $m+1$ variables. Furthermore, the properties of each constructed solution are investigated.

Key Words: Appell and Lauricella hypergeometric functions of variables, Generalized Euler-Poisson-Darboux Equation.

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INTRODUCTION

Euler-Poisson- Darboux Equation appears in various fields of mathematics and physics, such as theory of surfaces (Darboux, 1972), the propagation of sound (Copson, 1975), the colliding gravitational fields (Hauser and Ernest, 1989) etc. The equation

$$\frac{\partial^2 u}{\partial t^2}+\frac{k}{t}\frac{\partial u}{\partial t}=\sum_{i=1}^{m}\frac{\partial^2 u}{\partial {x_i}^2}+\frac{\lambda_i}{{x_i}^2}u \qquad (1.1)$$

Subject to the initial conditions

$$u(x,0)=f(x),\ \ u_t(x,0)=0 \qquad (1.2)$$

Was studied by Appell and Kampé (1926) in which $x=(x_1,x_2,\cdots\cdots,x_m)$. He determined the solution and investigated the Huygen's principle for (1.1)-(1.2). For $\lambda_i=0$ for every i (1.1) reduces to EPD which has been investigated extensively, since the appearance of Weinstein (1952). The author Seilkhanova and Hasanov (2015) constructed solutions of EPD in two dimensions in an explicit form expressed by Lauricella hypergeometric functions of three variables.

SOLUTIONS IN THE APPELL HYPERGEOMETRIC FORM

We consider the case (1.1) when $\lambda_i=0,\ i=2,3,\cdots m$, Equation (1.1) then reduces to the EPD equation

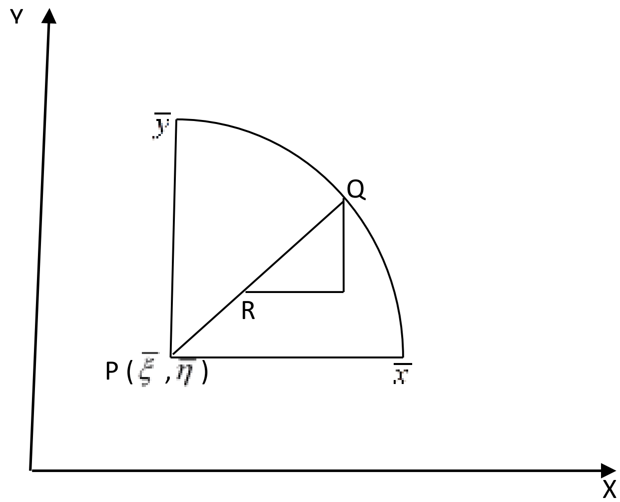

Figure 1. Geometry of the solution of Cauchy's problem.

$$\frac{\partial^2 u}{\partial t^2} + \frac{k}{t}\frac{\partial u}{\partial t} = \sum_{i=1}^{m} \frac{\partial^2 u}{\partial x_i^2} \qquad (2.1)$$

This reduces to

$$u_{XY} + \frac{N}{X+Y}\left(u_X + u_Y\right) - \frac{N}{X-Y}\left(u_X - u_Y\right) = 0 \qquad (2.2)$$

Where $X = r + t,\ Y = r - t,\quad r = \left(\sum_{i=1}^{m} x_i^2\right)^{\frac{1}{2}}$

Further, let $\xi = X^2, \quad \eta = Y^2$ then Equation (2.2) becomes

$$\frac{\partial^2 u}{\partial \xi \partial \eta} - \frac{\frac{k}{2}}{(\xi - \eta)}\left(\frac{\partial u}{\partial \xi} - \frac{\partial u}{\partial \eta}\right) = 0 \qquad (2.3)$$

Where $k = 2N$.

We note that $u = (\xi - a)^{\frac{-k}{2}}(\eta - a)^{\frac{-k}{2}}$ solves Equation (2.3), where a is a parameter. Let us now confine our attention to the half plane lying under the line $\xi = \eta$ (Figure 1) Take $\xi > \eta$, with the restrictions

$$\xi > \bar{\xi} > 0,\ \ \eta < \bar{\eta},\ \ \frac{k}{2} = \lambda \ \ (0<\lambda<1) \qquad (2.4)$$

We note that the adjoint of Equation (2.2) is given by

$$v_{\xi\eta} + \frac{\lambda - 1}{\xi - \eta}(v_\xi - v_\eta) = 0 \qquad (2.5)$$

Whose solution (Wanjala et al., 2012) is given by

$$v = \int_{\eta}^{\xi} \phi(a)(\xi - a)^{-\lambda}(a - \eta)^{-\lambda}\, da$$

Where ϕ is an arbitrary analytic function v is called the Riemann Greens function of (2.3). Taking $\phi(a) = 0$ for $\bar{\eta} \le a \le \bar{\xi}$, we obtain a two parameter family of solutions

$$v = \int_{\bar{\xi}}^{\xi} \phi(a)(\xi - a)^{-\lambda}(a - \eta)^{-\lambda}\, da + \int_{\eta}^{\bar{\eta}} \phi(a)(\xi - a)^{-\lambda}(a - \eta)^{-\lambda}\, da \qquad (2.6)$$

To determine the arbitrary function $\phi(a)$, we use the following definitions, in regard to the first integral of (2.6):

$$\begin{aligned} &\rho(t) = G(t) = 0, \quad 0 \le t \le \bar{\xi} \\ &\rho(t) = \phi(t)(t - \bar{\eta})^{-\lambda}(\xi - \eta)^{-\lambda}(\xi - \bar{\xi}) \\ &G(t) = t - \bar{\xi},\ t \ge \bar{\xi} \end{aligned} \qquad (2.7)$$

By use of Abel integral equation

$f(x)=\int_a^x \frac{u(\xi)}{(x-\xi)^{\lambda}},\ \ 0<\lambda<1$, whose solution is

$u(x)=\frac{\sin\pi\lambda}{\pi}\frac{d}{dx}\int_a^x \frac{f(t)}{(x-t)^{1-\lambda}}dt,$ We write

$\int_0^r \frac{\rho(t)}{(\xi-t)^{\lambda}}dt=G(r,\xi)$

Hence

$$\rho(\mathrm{a})=\frac{\sin\pi\lambda}{\pi}\int_0^a G'(\mathrm{t})(\mathrm{a-t})^{\lambda-1}dt=\frac{\sin\pi\lambda}{\pi}\int_{\xi}^a (a-t)^{\lambda-1}dt=\frac{\sin\pi\lambda}{\lambda\pi}(\bar{\xi}-t)^{\lambda}$$

From the second Equation of (2.7) we obtain

$$\phi(a)=\frac{\sin\pi\lambda}{\pi\lambda}\frac{(\mathrm{a}-\bar{\xi})^{\lambda}(a-\bar{\eta})^{\lambda}(\xi-\eta)^{\lambda}}{\xi-\bar{\xi}} \qquad (2.8)$$

With the substitution

$$a=\bar{\xi}+t(\xi-\bar{\xi}),\ a=\bar{\eta}+t(\eta-\bar{\eta})$$

In the first and second integrals of (2.6), respectively we obtain

$$v=\frac{\sin\pi\lambda}{\lambda\pi}\left\{(\xi-\eta)^{\lambda}\frac{\int_0^1 t^{\lambda}(\bar{\xi}-\bar{\eta}+t(\xi-\bar{\xi})^{\lambda}dt}{(1-t)^{\lambda}(\bar{\xi}-\eta+t(\xi-\bar{\xi}))^{\lambda}}+(\xi-\eta)^{\lambda}\frac{\int_0^1 t^{\lambda}(\bar{\xi}-\bar{\eta}-t(\eta-\bar{\eta}))^{\lambda}dt}{(1-t)^{\lambda}(\xi-\bar{\eta}-t(\eta-\bar{\eta}))^{\lambda}}\right\} \qquad (2.9)$$

Now if we let $x=\frac{\xi-\bar{\xi}}{\bar{\eta}-\xi},\ y=\frac{\xi-\bar{\xi}}{\eta-\bar{\xi}}$ in the first integral and $\bar{x}=\frac{\eta-\bar{\eta}}{\bar{\xi}-\bar{\eta}},\ \bar{y}=\frac{\eta-\bar{\eta}}{\xi-\bar{\eta}}$ in the second integral, then Equation (2.9) reduces to

$$\frac{\sin\pi\lambda}{\pi\lambda}\left[\left\{\left(\frac{\bar{\xi}-\bar{\eta}}{\bar{\xi}-\eta}\right)^{\lambda}(\xi-\eta)^{\lambda}\int_0^1 t^{\alpha-1}(1-t)^{\gamma-\alpha-1}(1-tx)^{-\beta}(1-ty)^{-\beta'}dt\right\}+\right.$$
$$\left.\left(\frac{\bar{\xi}-\bar{\eta}}{\xi-\bar{\eta}}\right)^{\lambda}(\xi-\eta)^{\lambda}\int_0^1 t^{\alpha-1}(1-t)^{\gamma-\alpha-1}(1-t\bar{x})^{-\beta}(1-t\bar{y})^{-\beta'}dt\right]$$

(2.10)

Since $\frac{\pi\lambda}{\sin\pi\lambda}=\frac{\Gamma(1+\lambda)\Gamma(1-\lambda)}{\Gamma(2)}$ the Equation (2.10) becomes

$$B(1+\lambda,1-\lambda)\left\{\left(\frac{\bar{\xi}-\bar{\eta}}{\bar{\xi}-\eta}\right)^{\lambda}(\xi-\eta)^{\lambda}F_1(1+\lambda;-\lambda,\lambda;2;x,y)+\right.$$
$$\left.\left(\frac{\bar{\xi}-\bar{\eta}}{\xi-\bar{\eta}}\right)^{\lambda}(\xi-\eta)^{\lambda}F_1(1+\lambda;-\lambda,2;\bar{x},\bar{y})\right\}$$

(2.11)

Where

$$F_1\left(a_1,\mathrm{b}_1,b_2,c;x,y\right)=\sum_{m,\mathrm{n}=0}^{\infty}\frac{(a)_{m+n}(b_1)_m(b_2)_n}{(c)_{m+n}m!n!}x^m y^n,$$

F_1 Being the Appell hypergeometric series defined for $|x|<1,\ |y|<1;$ and $(q)_n$ is the Pochhammer symbol representing the rising factorial: $(q)_n=\frac{\Gamma(q+n)}{\Gamma(q)}=q(q+1)\ldots\ldots(q+n-1).$

RECURRENCERELATIONSOF APPELL'S HYPERGEOMETRIC FUNCTIONS

Appell hypergeometric function can be written in two forms, namely: $F_1(\alpha,\beta,\gamma,\delta;x,y)$ and $F_2(\alpha,\beta,\gamma,\delta,\rho;x,y),$ which hold when $\max\{|x|,|y|<1\}$ and $|x|+|y|<1,$ respectively. Their corresponding integral representations are defined by the following

$$F_1(\alpha,\beta,\gamma,\delta;x,y)=\frac{\Gamma(\alpha)\Gamma(\gamma-\alpha)}{\Gamma(\gamma)}\int_0^1 t^{\alpha-1}(1-t)^{\delta-\alpha-1}(1-xt)^{-\beta}(1-yt)^{-\gamma}dt \qquad (3.1)$$

Where $\max\ \{|x|,|\mathrm{y}|<1\}$, and

$$F_2(\alpha,\beta,\gamma;\delta,\rho;x,y)=\frac{\Gamma(\delta)\Gamma(\rho)}{\Gamma(\beta)\Gamma(\delta-\beta)\Gamma(\gamma)\Gamma(\rho-\gamma)}\int_0^1\int_0^1\frac{t^{\beta-1}(1-t)^{\delta-\beta-1}s^{\gamma-1}(1-s)^{\rho-\gamma-1}}{(1-xt-ys)^{\alpha}}dtds.$$

(3.2)

Where

$|\mathrm{x}|+|\mathrm{y}|<1$ and $R(\delta)>R(\beta)>0,\ R(\rho)>R(\gamma)>0.$ using these integral representations, the following relations hold.

Theorem 3.1

For $R(\delta)>R(\beta)>0,\ R(\rho)>R(\gamma)>0$,

$$F_1(\alpha,\beta,\gamma;\delta;x,y)=(1-x)^{-\beta}(1-y)^{-\gamma}F_1(\alpha,\beta,\gamma;\delta;\frac{x}{x-1},\frac{y}{y-1})$$

Where $|\mathrm{x}|,|\mathrm{y}|<1$.

Proof

We let $t = 1 - s.$ then the right hand side of (3.1) becomes

$$\frac{\Gamma(\alpha)\Gamma(\gamma-\alpha)}{\Gamma(\gamma)}\int_0^1 s^{\delta-\alpha-1}(1-s)^{\alpha-1}(1-x(1-s))^{-\beta}(1-y(1-s))^{-\gamma}\,ds \qquad (3.3)$$

Note that $1 - x + xs = (1-x)(1-\frac{xs}{x-1}).$ Hence (3.3) reduces to

$$\frac{\Gamma(\alpha)\Gamma(\gamma-\alpha)}{\Gamma(\alpha)}(1-x)^{-\beta}(1-y)^{\gamma}\int_0^1 s^{\delta-\alpha-1}(1-s)^{\alpha-1}(1-\xi s)^{-\beta}(1-\eta s)^{-\gamma}\,ds$$

$$= (1-x)^{-\beta}(1-y)^{\gamma}F_1(\delta-\alpha,\beta,\gamma;\delta;\xi,\eta),$$

Where $\xi = \frac{x}{x-1}$ and $\eta = \frac{y}{y-1}.$

Theorem 3.2

For $R(\delta) > R(\beta) > 0,\ R(\rho) > R(\gamma) > 0$,

$$F_2(\alpha,\beta,\gamma;\delta,\rho;x,y) = (1-x)^{-\alpha}F_2(\alpha,\delta-\beta,\gamma;\delta,\rho;\frac{x}{x-1},\frac{-y}{x-1})$$

$$= (1-y)^{-\alpha}F_2(\alpha,\beta,\rho-\gamma;\delta,\rho;\frac{-x}{y-1},\frac{y}{y-1})$$

$$= (1-x-y)^{-\alpha}F_2(\alpha,\delta-\beta,\rho-\gamma;\delta,\rho;\frac{-x}{1-x-y},\frac{-y}{1-x-y})$$

Where $|x|+|y|<1$.

Proof

With the transformations

i.) $t = 1 - t',\ \ s = s';$

ii.) $t = t',\ s = 1 - s';$

iii.) $t = 1 - t',\ s = 1 - s'$

The proof is immediate for the first two cases. We discuss the third case: By substituting (iii) in the right hand side of Equation (3.2) we obtain

$$\frac{\Gamma(\delta)\Gamma(\rho)}{\Gamma(\beta)\Gamma(\delta-\beta)\Gamma(\gamma)\Gamma(\rho-\gamma)}\int_0^1\int_0^1\frac{(1-t')^{\beta-1}(t')^{\delta-\beta-1}(1-s')^{\gamma-1}(s')^{\rho-\gamma-1}}{(1-x+xt'-s+ys')^{\alpha}}dt'ds'$$

$$= \frac{\Gamma(\delta)\Gamma(\rho)}{\Gamma(\beta)\Gamma(\delta-\beta)\Gamma(\gamma)\Gamma(\rho-\gamma)}(1-x-y)^{-\alpha}\int_0^1\int_0^1(t')^{\delta-\beta-1}(1-t')^{\beta-1}(s')^{\rho-\gamma-1}(1-s')^{\gamma-1}(1-\frac{xt'-ys'}{1-x-y})^{-\alpha}ds'dt'$$

$$= \frac{\Gamma(\delta)\Gamma\rho)}{\Gamma(\beta)\Gamma(\delta-\beta)\Gamma(\gamma)\Gamma(\rho-\gamma)}(1-x-y)^{-\alpha}F_2(\alpha,\delta-\beta,\rho-\gamma;\delta,\rho;\frac{-x}{1-x-y}{,}\frac{-y}{1-x-y})$$

Recursion Formulas For F_1 and F_2 In Terms Of $_2F_1$.

The functions $F_1(x,\mathrm{y})$ and $F_2(x,y)$ satisfy the following systems of differential equations:

$$x(1-x)\frac{\partial^2 F_1}{\partial x^2} - y^2\frac{\partial^2 F_1}{\partial y^2} - 2xy\frac{\partial^2 F_1}{\partial x\partial y} + (\gamma-(\alpha+\beta+1)x)\frac{\partial F_1}{\partial x} - (\alpha+\beta+1)y\frac{\partial F_1}{\partial y} - \alpha\beta F_1 = 0$$

(4.1)

$$y(1-y)\frac{\partial^2 F_1}{\partial y^2} - x^2\frac{\partial^2 F_1}{\partial x^2} - 2xy\frac{\partial^2 F_1}{\partial x\partial y} + (\delta-(\alpha+\beta+1)y)\frac{\partial F_1}{\partial y} - (\alpha+\beta+1)x\frac{\partial F_1}{\partial x} - \alpha\beta F_1 = 0$$

And

$$x(1-x)\frac{\partial^2 F_2}{\partial x^2} - xy\frac{\partial^2 F_2}{\partial x\partial y} + (\delta-(\alpha+\beta+1))x\frac{\partial F_2}{\partial x} - \beta y\frac{\partial F_2}{\partial y} - \alpha\beta F_2 = 0$$

$$y(1-y)\frac{\partial^2 F_2}{\partial y^2} - xy\frac{\partial^2 F_2}{\partial x\partial y} + (\rho-(\alpha+\gamma+1))y\frac{\partial F_2}{\partial y} - \gamma x\frac{\partial F_2}{\partial x} - \alpha\gamma F_2 = 0$$

Respectively.

Theorem 4.1

The following pairs of functions:

1. $F_2(\alpha,\beta,\gamma;2\beta,2\gamma;x,2-x)$ *and*

$(x-2)^{-\alpha}\ {}_2F_1(\frac{\alpha}{2},\frac{\alpha+1}{2}-\gamma;\beta+\frac{1}{2};\frac{x^2}{(2-x)^2})$

2 . $F_2(\beta+\gamma-\frac{1}{2},\beta,\gamma;2\beta,2\gamma;\frac{4t}{(t-1)^2},\frac{(1-s)(st^2-1)}{s(t-1)^2})$

and $(1-t)^{2\beta+2\gamma-1}\ {}_2F_1(\beta+\gamma-\frac{1}{2},\gamma;\beta+\frac{1}{2};st^2)$, where s is an arbitrary constant, satisfy the second order ordinary differential Equations (4.2).

Proof

For the first case, we refer to (Vidunas, 2009). In the second case of the theorem, we substitute

$s \mapsto s^2,\ t \mapsto \frac{t}{s}$

And conclude that

$$F_2(\beta+\gamma-\frac{1}{2},\beta,\gamma;2\beta,2\gamma;\frac{-4st}{(t-s)^2},\frac{-(s^2-1)(t^2-1)}{(t-s)^2}) \qquad (4.3)$$

And

$$(s-t)^{2\beta+2\gamma-1}\,{}_2F_1(\beta+\gamma-\frac{1}{2},\gamma;\beta+\frac{1}{2};s^2)\,{}_2F_1(\beta+\gamma-\frac{1}{2},\gamma;\beta+\frac{1}{2};t^2) \quad (4.4)$$

Satisfy the same second order ordinary differential equations with respect to $t.$ In this setting ${}_2F_1(\beta+\gamma-\frac{1}{2},\gamma;\beta+\frac{1}{2};s^2)$ is a constant factor. However, the symmetry between x and y suggests that the system of Equations (4.2) for $F_2(\beta+\gamma-\frac{1}{2},\beta,\gamma;2\beta,2\gamma;\mathrm{x},\mathrm{y})$ can be transformed following (4.3) to a system of differential equations where the new variables s and t are separated. Using the substitution

$$s \mapsto \frac{s+1}{s-1}$$

In (4.3)-(4.4) and considering F_2 and ${}_2F_1$ solutions of the same system of differential equations around the point $(t,s)=(0,0),$ we have

$$F_2(\beta+\gamma-\frac{1}{2},\beta,\gamma;2\beta,2\gamma;\frac{4(1-s^2)t}{(1+s+t-st)^2},\frac{4(1-t^2)s}{(1+s+t-st)^2})$$
$$=\left(\frac{1+s+t-st}{1-s}\right)^{2\beta+2\gamma-1}{}_2F_1\left(\beta+\gamma-\frac{1}{2},\gamma;2\gamma;\frac{4s}{s^2-1}\right){}_2F_1\left(\beta+\gamma-\frac{1}{2},\gamma;\beta+\frac{1}{2};t^2\right) \quad (4.5)$$

Theorem 2.4

The functions

$F_1\left(\alpha,\beta;\gamma,\alpha+\beta-\gamma+\frac{3}{2};x^2,(1-x^2)\right)$ And ${}_2F_1\left(2\alpha,2\beta;2\gamma-1;x\right)$ Satisfy the system of differential Equations (4.1).

Proof

See Bailey (1933) where separation of variables for

$$F_1\left(\alpha,\beta;\gamma,\alpha+\beta-\gamma+1;\mathrm{x}(1-\mathrm{y}),\mathrm{y}(1-\mathrm{x})\right)$$
$$={}_2F_1\left(\alpha,\beta;\gamma;\mathrm{x}\right){}_2F_1\left(\alpha,\beta;\alpha+\beta-\gamma+1;y\right)$$

Via the transformation

$$F_1\left(\frac{\alpha}{2},\frac{\alpha+1}{2};\beta+\frac{1}{2},\gamma+\frac{1}{2};x^2,y^2\right)=\left(1+x+y\right)^{-\alpha}F_2\left(\alpha,\beta,\gamma;2\beta,2\gamma;\frac{2x}{x+y+1},\frac{2y}{x+y+1}\right).$$

Solution of (1.1) Where $\lambda_i \neq 0,\ 0<i<m.$

The author Fox (1959) determined the solution of (1.1)-(1.2) as

$$u(x,t,f)=\frac{\Gamma\left(\frac{1}{2}(k+1)\right)}{\pi^{\frac{m}{2}}\Gamma\left(\frac{1}{2}(k-m+1)\right)}\int_{S(x,t)} f(\xi)v(x,t,\xi)d\xi \quad (5.1)$$

in which $d\xi=d\xi_1 d\xi_2\ldots\ldots d\xi_m$ and $S(x,t)$ is the domain of the singular hyperplane $t=0$ cut out by the characteristic cone with vertex at (x,t) and where

$v=|t|^{1-k}\ \Upsilon^{\frac{1}{2}(k-m+1)}F_A\left(\alpha,1-\alpha,\frac{1}{2}(\mathrm{k}-m+1),z\right),$ Where Υ is given by the formula

$$\Upsilon(x,t;\xi)=t^2-\sum_{i=1}^{m}\left(x_i-\xi_i\right)^2,$$

$\alpha=(\alpha_1,\alpha_2,\ldots\ldots\alpha_m),\ 1-\alpha=(1-\alpha_1,1-\alpha_2,\ldots\ldots,1-\alpha_m)$ and F_A is a Lauricella hypergeometric function defined by

$$F_A(\alpha_1,\alpha_2,\ldots\ldots,\alpha_m;\beta_1,\beta_2,\ldots\ldots,\beta_m;\gamma;\mathrm{z}_1,\mathrm{z}_2,\ldots\ldots,\mathrm{z}_m)$$
$$=\sum_{\rho_1,\rho_2,\ldots\ldots,\rho_m=0}^{\infty}\frac{1}{\left(\gamma,\rho_1+\rho_2+\ldots\ldots+\rho_m\right)}\prod_{i=1}^{m}\frac{(\alpha_i,\rho_i)(\beta_i,\rho_i)}{\rho_i!}z_i^{\rho_i},$$

Where

$$(\alpha)_\rho=\frac{\Gamma(\alpha+\rho)}{\Gamma(\alpha)} \text{ and } \alpha_i=\frac{1+\sqrt{1-4\lambda_i}}{2},\ \beta_i=\frac{1-\sqrt{1-4\lambda_i}}{2},\ \gamma=\frac{1}{2}(k-m+1).$$

Taking the transformation $u=vx_i^{\alpha},$ (1.1) reduces to an equation of the form

$$\frac{\partial^2 v}{\partial t^2}+\frac{2\alpha_0}{t}\frac{\partial v}{\partial t}=\sum_{i=1}^{m}\left(\frac{\partial^2 v}{\partial x_i^2}+\frac{2\alpha_i}{x_i}\frac{\partial v}{\partial x_i}\right)$$

(5.2)

We discuss the particular case of Equation (5.2) when $m=3,$ which gives a generic solution of it. Let $u=\left(r^2\right)^{-\alpha-\beta-\gamma-\delta-\frac{1}{2}}w(\xi,\eta,\rho,\delta),$ r is the Lorentz distance between $(\mathbf{x},t)$ and a point $\boldsymbol{\xi},$ in the singular plane, given explicitly by

$$r(x_1, x_2, \ldots\ldots, x_m; t) = \sum (x_i - \xi_i)^2 - (t - t_0)^2.$$

Further,

$$\xi = \frac{r^2 - r_1^2}{r^2},\ \eta = \frac{r^2 - r_2^2}{r^2},\ \rho = \frac{r^2 - r_3^2}{r^2},\ \sigma = \frac{r^2 - r_4^2}{r^2}$$

And where

$$r_1^2 = (x_1 + \xi_1)^2 + (x_2 - \xi_2)^2 + \ldots\ldots + (x_m - \xi_m)^2 - (t - t_0)^2,$$

$$r_2^2 = (x_1 - \xi_1)^2 + (x_2 + \xi_2)^2 + \ldots\ldots + (x_m - \xi_m)^2 - (t - t_0)^2,$$

$$\vdots$$

$$\vdots$$

$$r_{m+1}^2 = (x_1 - \xi_1)^2 + (x_2 + \xi_2)^2 + \ldots\ldots + (x_m - \xi_m)^2 - (t + t_0)^2.$$

With $m = 3$, Equation (5.2) becomes

$$-4(r^2)^{-\alpha-\beta-\gamma-\delta-\frac{3}{2}} \left\{ \frac{A}{x} + \frac{B}{y} + \frac{C}{z} + \frac{D}{t} \right\} = 0 \qquad (5.3)$$

Where A, B, C and D are given as:

$$A = \xi(1-\xi)w_{\xi\xi} - \xi\eta w_{\xi\eta} - \xi\rho w_{\xi\rho} - \xi\sigma w_{\xi\sigma} + (2\alpha - (2\alpha + \beta + \gamma + \delta + \frac{3}{2})\xi)w_\xi$$

$$+\alpha\eta w_\eta + \alpha\rho w_\rho + \alpha\sigma w_\sigma - (\alpha + \beta + \gamma + \delta + \frac{1}{2})\alpha w,$$

$$B = \eta(1-\eta)w_{\eta\eta} - \xi\eta w_{\xi\eta} - \eta\rho w_{\eta\rho} - \eta\sigma w_{\eta\sigma} + \beta\xi w_\xi + (2\beta - (\alpha + 2\beta + \gamma + \delta + \frac{3}{2})\eta)w_\eta$$

$$+\beta\rho w_\rho + \beta\sigma w_\sigma + (\alpha + \beta + \gamma + \delta + \frac{1}{2})\beta w,$$

$$C = \rho(1-\rho)w_{\rho\rho} - \xi\rho w_{\xi\rho} - \eta\rho w_{\eta\rho} - \rho\sigma w_{\rho\sigma} + \gamma\xi w_\xi + \gamma\eta w_\eta + (2\gamma - (\alpha + \beta + 2\gamma + \delta + \frac{3}{2})\rho)w_\rho$$

$$+\delta\sigma w_\sigma + (\alpha + \beta + \gamma + \delta + \frac{1}{2})\gamma w,$$

$$D = \sigma(1-\sigma)w_{\sigma\sigma} - \xi\sigma w_{\xi\sigma} - \eta\sigma w_{\eta\sigma} - \rho\sigma w_{\rho\sigma} + \sigma\xi w_\xi + \delta\eta w_\eta + \delta\rho w_\rho + (2\delta - (\alpha + \beta + \gamma + 2\delta + \frac{3}{2})\sigma)w_\sigma$$

$$+(\alpha + \beta + \gamma + \delta + \frac{1}{2})\delta w.$$

Equation (5.3) holds true when $A = B = C = D = 0.$ thus the EPD Equation (5.2) reduces to a system of Lauricella hypergeometric equations; $A = 0,\ B = 0,\ C = 0,$ and $D = 0.$ the author Appell and Kampé (1926) considered the general case

$$x_j(1-x_j)\frac{\partial^2 F_A}{\partial x_j^2} - x_j \sum_{k=1,k\neq j}^{n} x_k \frac{\partial^2 F_A}{\partial x_k \partial x_j} + (c_j - (a + b_j + 1)x_j \frac{\partial F_A}{\partial x_j} - b_j \sum_{k=1,k\neq j}^{n} x_k \frac{\partial F_A}{\partial x_k} - ab_j F_A = 0,$$

$$j = 1, 2, \ldots\ldots, n$$

Where he found 2^n particular solutions of this system. All of them are expressed by Lauricella hypergeometric function F_A, whose integral representation is given by

$$F_A(\alpha, \beta_1, \beta_2, \ldots\ldots, \beta_m; \gamma_1, \gamma_2, \ldots\ldots, \gamma_m; x_1, x_2, \ldots\ldots, x_m)$$

$$= \prod_{j=1}^{m} B(\beta_j, \gamma_j - \beta_j) \int_0^1 \ldots\ldots \int_0^1 \prod_{j=1}^{m} u_j^{\beta_j - 1}(1-u_j)^{\gamma_j - \beta_j - 1} \left(1 - x_1u_1 - x_2u_2 \ldots\ldots - x_mu_m\right)^{-\alpha} du_1du_2 \ldots\ldots du_m,$$

$$R(\beta_j) > 0,\ R(\gamma_j - \beta_j) > 0,\ \ j = 1, 2, \ldots\ldots, m.$$

The series representation of F_A defined by Appell and Kampé (1926) is

$$F_A(\alpha, \beta_1, \beta_2, \ldots\ldots, \beta_m; \gamma_1, \gamma_2, \ldots\ldots, \gamma_m; x_1, x_2, \ldots\ldots, x_m)$$

$$= \sum_{n_1, n_2 \ldots\ldots n_m = 0}^{\infty} \frac{(\alpha)_{n_1 + n_2 + \ldots\ldots + n_m} (\beta)_{n_1}, (\beta)_{n_2}, \ldots\ldots (\beta)_{n_m}}{(\gamma)_{n_1} (\gamma)_{n_2} \ldots\ldots (\gamma)_{n_m}} \frac{x_1^{n_1} x_2^{n_2} \ldots\ldots x_m^{n_m}}{n_1! n_2! \ldots\ldots n_m!},$$

Valid for $|x_1| + |x_2| + \ldots\ldots + |x_m| < 1.$

PROPERTIES OF LAURICELLAHYPEGEOMETRIC FUNCTION

Lauricella hypergeometric function satisfies the following properties.

Theorem 6.1

The Lauricella hypergeometric function F_A satisfies

$$F_A(\alpha, \beta_1, \beta_2, \ldots\ldots, \beta_m; \gamma_1, \gamma_2, \ldots\ldots, \gamma_m; x_1, x_2, \ldots\ldots, x_m)$$

$$= \prod_{j=1}^{m} B(\beta_j, \gamma_j - \beta_j) \int_0^1 \ldots\ldots \int_0^1 \prod_{j=1}^{m} u_j^{\beta_j - 1}(1-u_j)^{\gamma_j - \beta_j - 1} \left(1 - x_1u_1 - x_2u_2 \ldots\ldots - x_mu_m\right)^{-\alpha} du_1du_2 \ldots\ldots du_m,$$

(6.1)

Where $R(\beta_i) > 0,\ R(\gamma_j - \beta_j) > 0,\ \ j = 1, 2, \ldots\ldots, m.;$ $|x_1| + |x_2| + \ldots\ldots + |x_m| < 1.$

Proof

In the proof of this Theorem, we apply the arguments suggested by Minjie (2013) Hassanov and Srivastava (2006). Here, we note that

$$(1-x_1u_1-x_2u_2-......-x_mu_m)^{-\alpha}=\sum_{n_1,n_2,......n_m=0}^{\infty}(\alpha)_{n_1+n_2+......+n_m}\frac{(x_1u_1)^{n_1}(x_2u_2)^{n_2}......(x_nu_m)^{n_m}}{n_1!n_2!......n_m!} \quad (6.2)$$

Substituting the right hand side of Equation (6.2) into (6.1) we obtain

$$F_A(\alpha,\beta_1,\beta_2,......,\beta_m;\gamma_1,\gamma_2,......,\gamma_m;x_1,x_2,......,x_m)$$
$$=(\prod_{j=1}^{m}B(\beta_j,\gamma_j-\beta_j))^{-1}\times$$
$$\int_0^1\int_0^1......\int_0^1\left(\prod_{j=1}^{m}u_j^{\beta_j-1}(1-u_j)^{\gamma_j-\beta_j-1}\sum_{n_1,n_2,......n_m=0}^{\infty}(\alpha)_{n_1+n_2+......+n_m}\frac{(x_1u_1)^{n_1}(x_2u_2)^{n_2}......(x_nu_n)^{n_m}}{n_1!n_2!......n_m!}\right)du_1du_2......du_m,$$
$$=\sum_{n_1,n_2,......nn_m=0}^{\infty}(\prod_{j=1}^{m}B(\beta_j,\gamma_j-\beta_j))^{-1}(\alpha)_{n_1+n_2+......+n_m}\frac{(x_1)^{n_1}(x_2)^{n_2}......(x_n)^{n_m}}{n_1!n_2!......n_m!}\int_0^1\int_0^1......\int_0^1\left(\prod_{j=1}^{m}u_j^{\beta_j-1}(1-u_j)^{\gamma j-\beta_j-1}du_j\right)$$
$$=\sum_{n_1,n_2,......n_m=0}^{\infty}(\alpha)_{n_1+n_2+......+n_m}\prod_{j=1}^{m}\frac{B(\beta_j+\eta_j,\gamma_j-\beta_j)}{B(\beta_j,\gamma_j-\beta_j)}\frac{(x_1)^{n_1}(x_2)^{n_2}......(x_n)^{n_m}}{n_1!n_2!......n_m!}$$

valid for $|x_1|+|x_2|+......+|x_m|<1$.

Theorem 6.2

For $R(\beta_i)>0,\ R(\gamma_j-\beta_j)>0,\ j=1,2,......,m.;$

$$F_A(\alpha,\beta_1,\beta_2,......,\beta_m;\gamma_1,\gamma_2,......,\gamma_m;x_1,x_2,......,x_m)$$
$$=\frac{1}{\Gamma(\alpha)}\int_0^1 e^{-t}t^{\alpha-1}\,{}_1F_1(\beta_1;\gamma_1;x_1t)\,{}_1F_1(\beta_2;\gamma_2;x_2t)......{}_1F_1(\beta_m;\gamma_m;x_mt)dt \quad (6.3)$$

Where $|x_1|+|x_2|+......+|x_m|<1.$

Proof

Recall that ${}_1F_1(a;b;z)=\sum_{k=0}^{\infty}\frac{(a)_k}{(b)_k}\frac{z^k}{k!}.$

The right hand side of (6.3) reduces to

$$\frac{1}{\Gamma(\alpha)}\int_0^1 e^{-t}t^{\alpha-1}\sum_{n_1,n_2,......n_m=0}^{\infty}\frac{(\beta_1)_{n_1}(\beta_2)_{n_2}......(\beta_m)_{n_m}}{(\gamma_1)_{n_1}(\gamma_2)_{n_2}......(\gamma_m)_{n_m}}\frac{x_1^{n_1}x_2^{n_2}......x_m^{n_m}}{n_1!n_2!......n_m!}dt$$
$$=\sum_{n_1,n_2,......n_m=0}^{\infty}\frac{(\beta_1)_{n_1}(\beta_2)_{n_2}......(\beta_m)_{n_m}}{(\gamma_1)_{n_1}(\gamma_2)_{n_2}......(\gamma_m)_{n_m}}\frac{x_1^{n_1}x_2^{n_2}......x_m^{n_m}}{n_1!n_2!......n_m!}\frac{1}{\Gamma(\alpha)}\int_0^1 e^{-t}t^{\alpha+n_1+n_2+......+n_m-1}dt$$
$$=\sum_{n_1,n_2,......n_m=0}^{\infty}\frac{(\beta_1)_{n_1}(\beta_2)_{n_2}......(\beta_m)_{n_m}}{(\gamma_1)_{n_1}(\gamma_2)_{n_2}......(\gamma_m)_{n_m}}\frac{x_1^{n_1}x_2^{n_2}......x_m^{n_m}}{n_1!n_2!......n_m!}\frac{1}{\Gamma(\alpha)}\int_0^1 e^{-t}t^{\alpha+n_1+n_2+......+n_m-1}dt$$
$$=\sum_{n_1,n_2,......n_m=0}^{\infty}(\alpha)_{n_1+n_2+......n_m}\frac{(\beta_1)_{n_1}(\beta_2)_{n_2}......(\beta_m)_{n_m}}{(\gamma_1)_{n_1}(\gamma_2)_{n_2}......(\gamma_m)_{n_m}}\frac{x_1^{n_1}x_2^{n_2}......x_m^{n_m}}{n_1!n_2!......n_m!}$$
$$=F_A(\alpha,\beta_1,\beta_2,...\beta_m;\gamma_1,\gamma_2...,\gamma_m;x_1,x_2,...,x_m)$$

Corollary 6.1

For $x_1=x_2=...=x_m=x$ *we have that*

$$F_A(\alpha,\beta_1,\beta_2,...,\beta_m;\gamma_1,\gamma_2,...,\gamma_m;x,x,...,x)=\sum_{n_1,n_2,...,n_m=0}^{\infty}(\alpha)_{n_1+n_2+...+n_m}\frac{(\beta_1)_{n_1}(\beta_2)_{n_2}...(\beta_m)_{n_m}}{(\gamma_1)_{n_1}(\gamma_2)_{n_2}...(\gamma_m)_{n_m}}\frac{x^{n_1+n_2+...+n_m}}{n_1!n_2!...n_m!}.$$

Also, if $x_1=x_2=...=x_m=1,$ *then*

$$F_A(\alpha,\beta_1,\beta_2,...,\beta_m;\gamma_1,\gamma_2,...,\gamma_m;1,1,...,1=\sum_{n_1,n_2,...,n_m=0}^{\infty}(\alpha)_{n_1+n_2+...+n_m}\frac{(\beta_1)_{n_1}(\beta_2)_{n_2}...(\beta_m)_{n_m}}{(\gamma_1)_{n_1}(\gamma_2)_{n_2}...(\gamma_m)_{n_m}}\frac{1}{n_1!n_2!...n_m!}.$$

Theorem 6.3

The following relation for F_A holds:

$$F_A(\alpha,\beta_1,\beta_2,...,\beta_m;\gamma_1,\gamma_2,...,\gamma_m;x_1,x_2,...,x_m)$$
$$=\sum_{n_1,n_2,...n_m=0}(\alpha)_{n_1+n_2+...+n_{m-1}}\prod_{j=1}^{m-1}\frac{B(\beta_j+\eta_j,\gamma_j-\beta_j)}{B(\beta_j,\gamma_j-\beta_j)}\frac{x_1^{n_1}x_2^{n_2}...x_{m-1}^{n_{m-1}}}{n_1!n_2!...n_{m-1}!}{}_2F_1(\alpha+n_1+n_2+...+n_{m-1},\beta_m;\gamma_m;x_m) \quad (6.4)$$

Proof

From the definition of F_A,

$$F_A(\alpha,\beta_1,\beta_2,...,\beta_m;\gamma_1,\gamma_2,...,\gamma_m;x_1,x_2,...,x_m)$$
$$=\sum_{n_1,n_2,...n_m=0}(\alpha)_{n_1+n_2+...+n_m}\prod_{j=1}^{m}\frac{B(\beta_j+\eta_j,\gamma_j-\beta_j)}{B(\beta_j,\gamma_j-\beta_j)}\frac{x_1^{n_1}x_2^{n_2}...x_{m-1}^{n_m}}{n_1!n_2!...n_m!},\ \text{where}\ |x_1|+|x_2|+...+|x_m|<1. \quad (6.5)$$

The right hand side of (6.5) becomes

$$F_A(\alpha,\beta_1,\beta_2,...,\beta_m;\gamma_1,\gamma_2,...,\gamma_m;x_1,x_2,...,x_m)$$
$$=\left(\sum_{n_1,n_2,...n_m=0}(\alpha)_{n_1+n_2+...+n_{m-1}}\prod_{j=1}^{m-1}\frac{B(\beta_j+\eta_j,\gamma_j-\beta_j)}{B(\beta_j,\gamma_j-\beta_j)}\frac{x_1^{n_1}x_2^{n_2}...x_{m-1}^{n_{m-1}}}{n_1!n_2!...n_{m-1}!}\right)\times$$
$$\left(\sum_{n_m=0}(\alpha+n_1+n_2+...+n_{m-1})_{n_m}\frac{B(\beta_{n_m}+\eta_m,\gamma_{n_m}-\beta_{n_m})}{B(\beta_{n_m},\gamma_{n_m}-\beta_{n_m})}\frac{x_m^{n_m}}{n_m!}\right),$$

This ends the proof.

REMARK

We observe from the proof of the above theorem that the Lauricella multivariable hypergeometric function can be decomposed into the products of the ordinary Gauss hypergeometric functions using the relation connecting the Pochhammer symbol $(\alpha)_{m_1+m_2+\ldots+m_r} = (\alpha)_{m_1+m_2+\ldots+m_{r-1}}(\alpha+m_1+m_2+\ldots+m_{r-1})_{m_r}$.

To this far, we have shown in general that the Riemann function of the adjoint Equation of (2.1) is given in terms of the Appell Hypergeometric function. Also we can comfortably draw the conclusion that since the solution of the generalized EPD Equation (1.1) is in terms of the Lauricella hypergeometric equation, then it is self adjoint.

Conclusion

We conclude this paper with a theorem:

Theorem 6.4

The solution of the problem (1.1) subject to the initial conditions (1.2) is given by

$$\left[(r+\bar{r})-(s+\bar{s})\right]u(\bar{p})v(\bar{p})=\frac{1}{2}\left\{(\bar{x}_r-\bar{x}_s)u(\bar{x})v(\bar{x})-(\bar{y}_r-\bar{y}_s)u(\bar{y})v(\bar{y})\right\}$$
$$-\int_{\bar{x}}^{\bar{y}}\left\{P(u,v)ds+Q(u,v)dx\right\}$$

Where $\bar{x}=(\bar{x}_r-\bar{x}_s),\bar{y}=(\bar{y}_r-\bar{y}_s)$

$$P(u,v)=\frac{1}{2}(r-s)(v\partial_s u=u\partial_s v)-\left(N-\frac{1}{2}\right)uv,$$

$$Q(u,v)=\frac{1}{2}(r-s)(v\partial_r u=u\partial_r v)-\left(N-\frac{1}{2}\right)uv,$$

We have taken recourse of green's theorem in the plane to write this solution where v is the Riemann function obtained in Equation (2.11) for the special case (2.1) and v is the Lauricella Hypergeometric equation F_A for the general case Equation (2.1). It is not hard to show that this solution satisfies the given equation with the corresponding initial conditions.

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